Analysis of nonlinear synchronization dynamics of oscillator networks by Laplacian spectral methods

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We analyze the synchronization dynamics of phase oscillators far from the synchronization manifold, including the onset of synchronization on scale-free networks with low and high clustering coefficients. We use normal coordinates and corresponding time-averaged velocities derived from the Laplacian matrix, which reflects the network's topology. In terms of these coordinates, synchronization manifests itself as a contraction of the dynamics onto progressively lower-dimensional submanifolds of phase space spanned by Laplacian eigenvectors with lower eigenvalues. Differences between high and low clustering networks can be correlated with features of the Laplacian spectrum. For example, the inhibition of full synchoronization at high clustering is associated with a group of low-lying modes that fail to lock even at strong coupling, while the advanced partial synchronization at low coupling noted elsewhere is associated with high-eigenvalue modes.

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The relation between structure and function is a key area in the study of complex networks [1–4]. Synchronization of coupled oscillators [5] has applications to numerous areas of biology including neuroscience, as well as systems such as coupled lasers and Josephson junctions, and accordingly its dependence on coupling topology has begun to receive attention. Among methods of studying synchronization, the master stability function (MSF) [6] formalism is appealing because it expresses the dynamical synchronizability in terms of purely structural features, independent of details of node dynamics. The so-called propensity for synchronization (an indication of the size of the parameter range giving a stable synchronized state) [7] depends only on the extremal eigenvalues of the Laplacian matrix. Within this formalism, the effects of small-world properties, heterogeneity, and certain types of weighted coupling have been examined [8,7]. The MSF, however, is restricted to the linear domain, close to exact amplitude and phase synchronization of chaotic oscillators. Others [9–13] have examined numerically and analytically the onset of synchronization for phase oscillators coupled on networks, a problem for which the MSF is unsuited.

In this Brief Report we demonstrate an application of the Laplacian spectrum to a sparsely connected network Kuramoto [14] model both close to and far from full synchronization. As a case study, we examine scale-free networks with low and high clustering coefficients, examined elsewhere by different methods [12]. Parametrizing the phase space with normal coordinates based on Laplacian eigenvectors, we show in these two sample cases that with increasing coupling strength, the dynamics contracts onto progressively lower-dimensional subspaces spanned by lower-lying (less stable) eigenvectors. Dynamical properties of the networks can be correlated with specific features of their spectra. By focusing on appropriately chosen collective degrees of freedom (the normal coordinates), our approach complements methods of analysis that focus on the locking and unlocking of individual oscillators [14,12]. In the spirit of the MSF, our analysis highlights the effects of network topology via the spectrum, but in contrast it applies to a range of desynchronized and partly synchronized states, not only to incipient deviations from full synchronization. We consider the spectrum in its entirety, not only the extremal eigenvalues. The coordinates derived from the Laplacian spectrum provide a helpful empirical tool for the analysis of simulation results. We use them here to gain new insight into the different behaviors of networks with high and low clustering coefficients. Our emphasis is on the process of synchronization, rather than on rigorous bounds for the threshold of desynchronization.

We first define the model and show how the Laplacian and its spectrum appear naturally in a linearized description of the frequency-synchronized state. Then we use the Laplacian eigenvectors to parametrize the partially desynchronized states and show that this coordinate system remains useful well beyond the range of validity of the linearization.

Our model [9,10] is defined by the coupled equations

$$\frac{d\phi_i}{dt} = \omega_i + \frac{\beta}{\langle k \rangle} \sum_j a_{ij} \sin(\phi_i - \phi_j), \tag{1}$$

where ϕ_i are N phase variables (one associated with each node of a network), $-1 \leqslant \omega_i \leqslant 1$ are the randomly and uniformly distributed intrinsic frequencies,[1] β is the overall coupling strength, and a_{ij} is the weighting matrix of the individual couplings. In our examples, all links are weighted equally, and a_{ij} is simply the adjacency matrix ($a_{ij}=1$ if i and j are connected, 0 otherwise). As in [10,12] the coupling strength is normalized by the average degree $\langle k \rangle$ of all nodes.[2] At low coupling strength, each oscillator moves independently at its intrinsic frequency, but as the coupling increases some become mutually entrained. At sufficiently strong coupling, all oscillators rotate at the same frequency, $\frac{d\phi_i}{dt} = \bar{\omega} = 0$. For the original, fully connected model the steady-state phases depend only on the intrinsic frequencies

[1]The average of ω_i can be taken to be $\bar{\omega}=0$ without loss of generality (if it is not zero it can be made so by changing variables into a rotating frame of reference).

[2]By normalizing the average total input to a unit this convention facilitates comparisons among networks and corresponds to the normalization by N in the original fully connected Kuramoto model.

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ω_i: those with higher frequencies lead the ensemble while those with lower frequencies lag. In the present sparsely connected version, on the other hand, each oscillator is influenced differently by its local neighborhood. If the phase differences are small then the sine coupling function can be approximated linearly and the equations of motion become

$$\frac{d\phi_i}{dt} = \omega_i + \frac{\beta}{\langle k\rangle}\sum_j a_{ij}(\phi_i - \phi_j) = \omega_i - \frac{\beta}{\langle k\rangle}\sum_j \pounds_{ij}\phi_j \qquad (2)$$

where

$$\pounds_{ij} = a_{ij} - \delta_{ij}\sum_k a_{ik} = a_{ij} - \delta_{ij}k_i \qquad (3)$$

is the Laplacian matrix. The degree k_i of the ith node is defined as the number of nodes to which it is connected, and the second equation in Eq. (3) thus holds if the couplings are equally weighted.

The steady-state (frequency locked) phases can be found by diagonalizing the Laplacian. Let its normalized eigenvectors and corresponding eigenvalues be $\mathbf{v}^\alpha$ and λ^α, where $1 \leqslant \alpha \leqslant N$. Enumerating lattice sites by Latin indices and Laplacian eigenvectors by Greek ones, we define projections of the phase and frequency vectors onto these eigenvectors by

$$\phi^\alpha \equiv \sum_i \phi_i v_i^\alpha, \quad \omega^\alpha \equiv \sum_i \omega_i v_i^\alpha, \qquad (4)$$

which allows the equations of motion (2) to be rewritten as

$$\frac{d\phi^\alpha}{dt} = \omega^\alpha - \frac{\beta}{\langle k\rangle}\lambda^\alpha\phi^\alpha. \qquad (5)$$

The steady state values $\overline{\phi^\alpha}$ of the normal coordinates are given by

$$\overline{\phi^\alpha} = \frac{\langle k\rangle}{\beta\lambda^\alpha}\omega^\alpha. \qquad (6)$$

Relaxation to this equilibrium obeys

$$\frac{dx^\alpha}{dt} = -\frac{\beta\lambda^\alpha}{\langle k\rangle}x^\alpha, \qquad (7)$$

where $x^\alpha = \phi^\alpha - \overline{\phi^\alpha}$ is the displacement from equilibrium along the αth normal coordinate. The equilibrium is stable provided all $\lambda^\alpha \geqslant 0$ and the phase displacements are small enough for the linear approximation to hold. By the definition (3), the row sum $\Sigma_j \pounds_{ij}$ of the Laplacian is zero for all rows and therefore (1,1,…1) is always an eigenvector with eigenvalue 0, but for a connected network, all other eigenvalues are positive [15]. Therefore the frequency synchronized state is neutrally stable against a uniform shift of all phases, but stable against all other perturbations. The stability breaks down only due to nonlinear effects: the slope of the sinusoidal coupling function decreases with increasing phase differences and eventually ceases to provide sufficient restoring force. Since the phase displacements are largest along the eigenvectors with lowest λ^α, these eigenvectors represent modes along which frequency synchronization first fails as the coupling decreases.

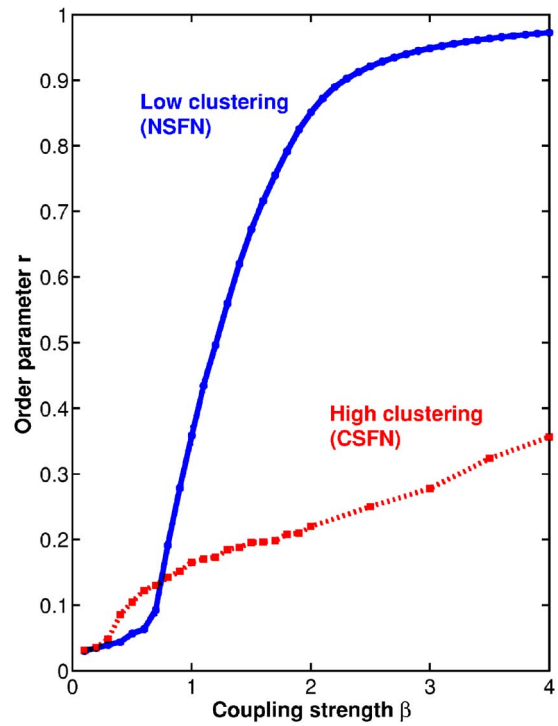

FIG. 1. (Color online) Synchronization order parameter as a function of coupling strength for the two scale-free networks. The strongly clustered network (CSFN) undergoes a partial synchronization at lower coupling strength, but at higher couplings it is significantly less synchronized than the normal (NSFN) network.

Although they arise most naturally from the linear analysis, the Laplacian eigenvectors retain their usefulness beyond that approximation. To demonstrate this, we consider two networks as examples. Our two networks have identical, scale-free, degree distributions but differ in their clustering coefficient [3,16]—a measure of the likelihood that two neighbors of a given node are also directly connected to each other, or a measure of the prevalence of triangles in the network topology. The first is a Barabasi-Albert [17] scale-free network of $N=1000$ nodes with average degree $\langle k\rangle = 20$, grown by means of preferential attachment beginning with a fully connected core of $m=10$ nodes. The Barabasi-Albert network has a low clustering coefficient, approximately 0.02. The other network is derived from the first by applying Kim's [18] stochastic rewiring method to increase the clustering coefficient to 0.62, without changing the degree distribution (although, as mentioned below, some other properties vary in tandem with the clustering). We will refer to these networks as the normal scale-free network (NSFN) and the clustered scale-free network (CSFN), respectively. These were among the networks studied previously in [12], where it was found that increased clustering inhibited full synchronization at high β but surprisingly promoted the onset of partial synchronization at low β. This behavior is shown in a plot (Fig. 1) of the standard synchronization order parameter

$$r = \left\langle \left| \sum_j e^{i\phi_j} \right| \right\rangle_T \qquad (8)$$

(where $\langle\cdots\rangle_T$ stands for a time average) as a function of the coupling strength. In the unsynchronized state at low cou-

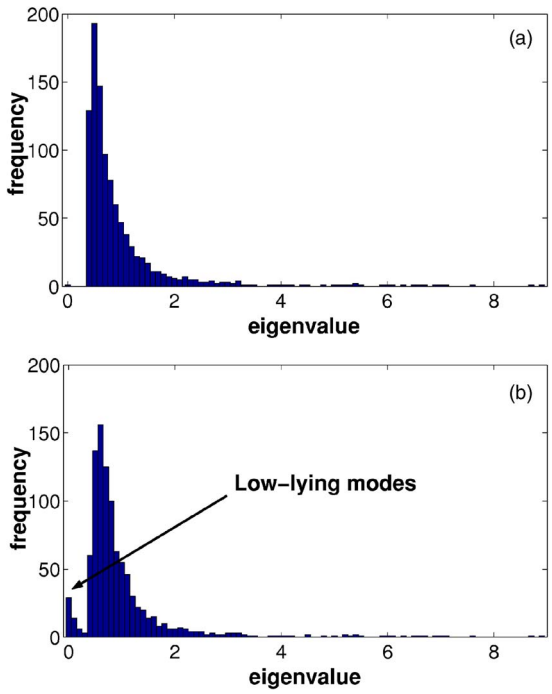

FIG. 2. (Color online) Histograms of the scaled Laplacian eigenvalues $\lambda^\alpha/\langle k\rangle$ for the NSFN (a) and CSFN (b). Both histograms have power-law tails at large eigenvalues. A key difference is the group of low-lying modes, separate from the main spectrum, in the highly clustered network.

pling, $r=O(1/\sqrt{N})$ from both networks. The onset of synchronization is shown by an upward turn in the plot of r vs β. This transition occurs at a lower β for the CSFN than for the NSFN, so that for $0.25\lesssim\beta\lesssim 0.75$, r is larger for the CSFN. At higher couplings, however, the CSFN strongly resists full synchronization and remains in a partly synchronized state with a much smaller value of r than for the NSFN.

The Laplacian eigenvalue spectra of the NSFN and CSFN are shown in Fig. 2. Like the degree distribution, the distribution of eigenvalues has a power-law tail in both cases. An important difference appears at the lower end of the spectrum. In the NSFN, there is a gap between the lowest nonzero eigenvalue and zero, and the single peak of the distribution is near this lower cutoff. The CSFN spectrum, on the other hand, has a second peak close to zero, indicating a number of nearly degenerate quasizero modes. The presence of eigenvalues close to zero indicates that the network has a strong community structure [20], i.e., it consists of components (communities) that have fewer connections between different components than within each component. In fact, the low eigenvectors of the Laplacian form the basis of some algorithms for detecting communities [19,20]. From the spectrum, then, we learn that the rewiring algorithm has not only created clustering (a local property measuring the number of triangles) but as a byproduct has also created global communities. Since higher clustering means more "local" connections at the expense of long-range ones, this is not surprising, but neither is it inevitable—for example, a regular ring or a "small-world" network of the type considered in [16] has high clustering but no communities.

To further aid in analyzing the dynamics we define the observed frequencies (rotation numbers) of the oscillators as the time averages

$$\Omega_j = \left\langle \frac{d\varphi_j}{dt} \right\rangle_T . \tag{9}$$

Projecting the vector of observed frequencies onto the Laplacian eigenbasis gives a time-averaged velocity along the direction defined by each eigenvector:

$$\Omega^\alpha = \sum_j \Omega_j v_j^\alpha . \tag{10}$$

In a fully frequency-synchronized state, $\Omega^\alpha=0$ for all α. In Fig. 3 ensemble averages of the squares of the velocities $[(\Omega^\alpha)^2]_\omega$ are plotted against the eigenvalues λ^α for both networks at three values of the coupling strength. The average $[\cdots]_\omega$ is over 25 different realizations of the random frequency distribution. In a case where the network is almost completely incoherent (for example, the NSFN at $\beta=0.5$), all

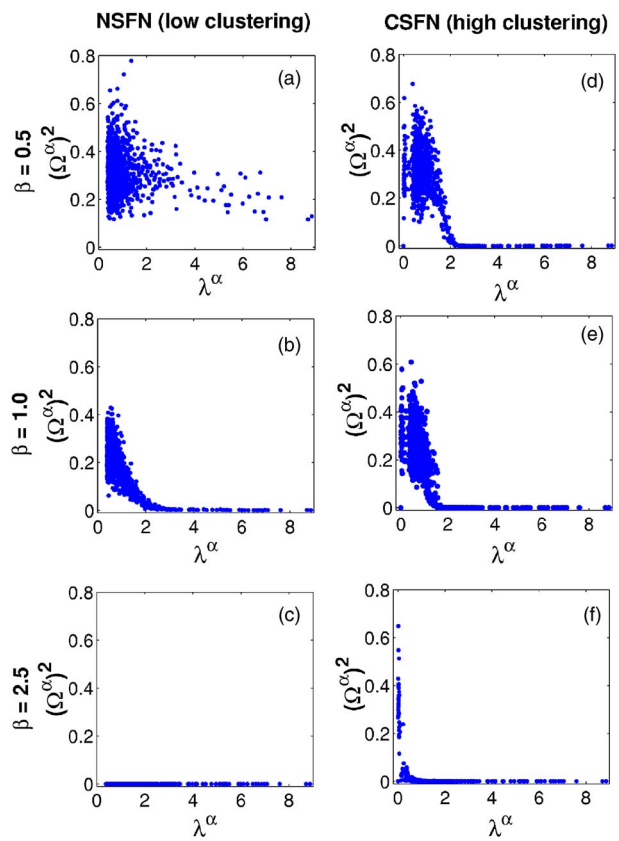

FIG. 3. (Color online) Mean-square projections of observed frequency differences onto Laplacian eigenvectors (normal velocities) at several values of coupling strength β. Each point represents an average over 25 realizations of the random intrinsic frequencies. In both types of networks, velocities along the eigenvectors with higher eigenvalues vanish at lower couplings than those with lower eigenvalues. In the highly clustered network (right column), two features are notable: The higher mode velocities vanish more readily than for the low-clustering network, and the lowest lying modes maintain nonzero velocities even at $\beta=2.5$ where all others vanish.

velocities Ω^{α} are random and of approximately equal magnitude. In the case of full synchronization (NSFN at β=2.5), all velocities are "locked" at zero. At intermediate values af β, however, some modes are locked while others are "drifting" at nonzero velocities. It is clear from the plots that as the coupling increases, modes with higher eigenvalues lock sooner than those with lower ones—synchronization proceeds from the top of the spectrum downward. Synchronization manifests itself as a progressive contraction of the dynamics onto lower-dimensional submanifolds of the phase space.

In the case of the CSFN, higher modes begin to lock more readily than in the NSFN, indicating that these high modes in the spectrum are implicated in the advanced partial synchronization of the clustered network (Fig. 1). At stronger coupling, on the other hand, the most notable difference of the clustered from the normal network is that the low-lying modes associated with community divisions (Fig. 2) continue to drift while all others are locked. The lack of synchronization is associated with these low-lying modes, and the frequency clusters noted in this case [12] coincide with topological communities. The observation that these low-lying modes fail to lock is consistent with our intuition based on the linear approximation, according to which these modes represent the strongest potential instabilities of a synchronized state. Their presence in the spectrum accounts for the inhibition of full synchronization in the CSFN. The finding that different sets of eigenvectors are involved in the two regimes supports the claim [13] that two separate effects are at work, with the advanced onset being an effect of the clustering *per se*, which is a local property, while the delay of full synchronization results from global topological properties that are correlated with clustering. In particular, the delay was ascribed to effects of increasing average path length [13]. However, the involvement of the low eigenvectors associated with communities and the dymanical fragmentation of the network into synchronized subgroups suggest that it is more specifically a function of the community structure (although the latter certainly is correlated with a long average path length). Ongoing studies aim to further disentangle the various correlated topological features and their effects.

Examining the dynamics in terms of normal coordinates defined by the Laplacian eigenvectors provides a geometric basis for viewing the flow of the ensemble of oscillators that complements other tools of analysis such as global order parameters [14] or scatter plots of observed vs intrinsic frequencies of individual oscillators [12]. Like the MSF formalism, it gives a partial picture of how purely structural features influence the synchronization dynamics, since the Laplacian reflects only the network topology. It is not obvious *a priori* that Laplacian eigenvectors should be relevant beyond the range of validity of the linear approximation near a fully phase-synchronized state, yet the normal coordinate velocities, in particular, contain nontrivial dynamical information well away from this limit, and they split into subsets associated with different dynamical effect. They allow one to indentify collective degrees of freedom responsible for on one hand the advanced partial synchronization and on the other the inhibition of complete synchronization in a highly clustered scale-free network. Connections among topology, spectrum, and dynamics will be explored more fully in a future publication, which will apply the formalism to other types of networks including ones with unequal and asymmetric couplings a_{ij}, as well as considering other spectral properties such as the localization and delocalization of modes.

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